Project I : Brownian motion in 1D

We consider the overdamped motion of a Brownian particle in one-dimensional space. The position of the particle is governed by the Langevin equation

$$\gamma\dot{x}(t) = f + \sqrt{2\gamma T}\xi(t) \text{ or } \gamma dx(t) = f dt + \sqrt{2\gamma T} dW(t). \tag{5.1}$$

The Brownian particle is driven by the constant force f. Initially, the Brownian particle is distributed according to the probability distribution function $P(x,0) = p_i(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2}$ with a constant σ.

[1] Find the probability distribution function $P(x,t) = P_f(x)$ at time t. You may use the solution of the Langevin equation or solve the Fokker-Planck equation directly.
[2] We denote the stochastic path during the time interval $0 \le t \le \tau$ by $[\boldsymbol{x}]_0^\tau$. Show that the total entropy production is given by the simple form

$$\Delta S([\boldsymbol{x}]_0^\tau) = -\ln \frac{P_f(x(\tau))}{P_i(x(0))} + \frac{f\{x(\tau) - x(0)\}}{T}. \tag{5.2}$$

[3] You can solve numerically the Langevin equation to generate a stochastic path $[\boldsymbol{x}]_0^\tau$ and the entropy production using (5.2). By repeating the simulation over and over, you can generate the probability distribution function $P(S)$ for the entropy production. Confirm the integral fluctuation theorem $\langle e^{-\Delta S} \rangle = 1$ at several different values of τ. All the other parameters γ, T, and f may be set to unity.