Project II : Rotary motor

The motion of a rotary motor is described by the overdamped Langevin equation

$$\gamma d\theta(t) = -\frac{dV(\theta)}{d\theta}dt + f_0 dt + \sqrt{2\gamma k_B T}dW(t) \tag{5.3}$$

where $0 \leq \theta < 2\pi$ is the angular position of the motor, $V(\theta) = \cos\theta$ is the periodic potential, and f_0 is a constant applied torque, and $W(t)$ denotes the Wiener process. One can set $\gamma = k_B = T = 1$.

[1] Write down the corresponding Fokker-Planck equation for the probability distribution function $P(\theta, t)$ and solve it to find the stationary state distribution $P_{ss}(\theta)$. Compare it with the equilibrium Boltzmann distribution $P_{eq}(\theta) = \frac{1}{Z}e^{-\beta V(\theta)}$ without torque.
[2] The average angular velocity $\Omega = \lim_{t\to\infty}\frac{1}{t}\langle(\theta(t) - \theta(0))\rangle$ depends on the applied torque f_0. Draw the plot of Ω as a function of f_0. This can be done analycally or numerically.
[3] Initially $f_0 = 0$ and the system is prepared in the equilibrium state with $P_{eq}(\theta)$. At time $t = 0$, the torque is turned to take the value $f_0 = 1$. Solve the Langevin equation numerically up to time $t = \tau$ and measure the work done by the external torque $W(\tau) = \int_0^\tau f_0\dot{\theta}(t')dt'$. By performing the numerical simulations many times, you can construct the probability distribution function for the work $P(W)$. Test the integral and detailed fluctuation theorems by computing $\langle e^{-W}\rangle$ and $P(W)/P(-W)$.